A Proof of Euclid's Fifth Axiom using Concepts of Elementary Calculus

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ISBN 978-0-473-31672-3

Author's Note:

In the following proof, reliance is placed upon the Leibniz derivative and I make a claim that popular calculus texts adhering to the canons of Zermelo-Fraenkel set theory do not properly define it. That being so, readers may discern that with changes, a similar proof could be given within the confines of the Zermelo-Fraenkel set theory.

There are no doubt many proofs of calculus that can be tackled, like the following one, in two different ways. The respective axiom systems have sufficient in common that they can both encapsulate the fundamental notions of calculus. Yet thinking through a proof can be quite a different experience if one is using the Leibniz derivative. A 'domain of form', for example, owns no parallel concept in the Zermelo-Fraenkel set theory because in the latter theory limits are defined only when their points of determination are explicitly referred to. On the other hand, in the context of the Leibniz derivative, such a concept enables one to place the point of determination within the graph of a given relation as opposed to outside the graph. That the point of determination may be outside is inevitable in certain situations, so a general account of the Leibniz derivative must allow for both inside and outside to occur.

Whereas mathematics education has become a must-have for most aspiring teenagers, we may hear a call for unification. Some observers of society at large will claim that if the thinking patterns of all citizens are unified under a common system, that will facilitate the advance of Man. However a contrary claim is also plausible. It may instead facilitate Man to fall into a hole. My need to go beyond the Zermelo-Fraenkel system was first felt while I was writing specifications for a patent application. Later in exploring the language of the law more fully, I became quite convinced that conceptual omissions germane to the Zermelo-Fraenkel theory could be preventing students from acquiring a full technical language as apropos regulation and law. It also appears that a lot of physics is written with the calculus of the Leibniz derivative. Therefore I recommend the following proof as an example of a way of thinking that may improve the lot of Man.

That Euclid's fifth axiom should apply to appropriately defined lines by no means rules out the possibility of space-time curvature. It may rather suggest an approach to the study of curvature, given a certain context.

Published in New Zealand by
Knüsel Information Tapui Ltd
79 Bridge St
Eltham 4322
www.skybicycle.biz

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Upon trying to specify a circle for the purposes of Euclid's third axiom, one may easily become embroiled in the question of the general validity of the theorem of pythagorus. The existence of a circle having a given centre {(a,b)} and a given radius r, may turn upon the existence of the following set: $\{(x,y): (x-a)^2 + (y-b)^2 = r^2\}$. The theorem of pythagorus would justify the view that this set is a circle, or rather that it passes on a circle, provided the variables a, b and r are independent of the bound variables x and y. The independence of the free variables from the bound variables in the notation for a set may be taken as a given, as generally understood unless especially excepted from. The theorem of pythagorus however depends on the fifth axiom and so a proof of the fifth axiom is called for.[1]

That the fifth axiom admits of the following proof is merely to show that Euclid's five axioms are encountered as an indivisible group, assuming reasonable definition is given to the term *angle* and terms associated with *angle*, and assuming straight lines are infinite linear lines in a sense we can define with some calculus.

My acquaintance with the requisite calculus has been enhanced somewhat by my discovery of a solution to Bertrand Russell's paradox in set theory. This has been written up in my book *Russell's Paradox - Two Sets*[2]. The book contains a definition of the limit and the derivative for elementary calculus. Based on these definitions, it is possible to give an exact formal account of what it means to assert that a derivative has constant form. Rather than preface the following proof with a discussion about this, in what follows I refer to the constant form without ado. As the concept is quite intuitive, this approach may have the advantage of whetting the reader's appetite for the discussion by first demonstrating the usefulness of the concept.

The notion of constant form for a derivative is also at the root of the idea of a line being linear. The definition of linearity and the discussion about constant form are therefore both left until after the proof has been given.

The following is Euclid's fifth[3] axiom:

> That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles. (Euclid)

We will begin by examining the three lines in a two-dimensional coordinate system wherein the x-axis is the line which "falls" on the two others. Let these others be {(x,y): A} and {(x,y): B} or simply A* and B* respectively. A and B are relations. The diagram overleaf applies.

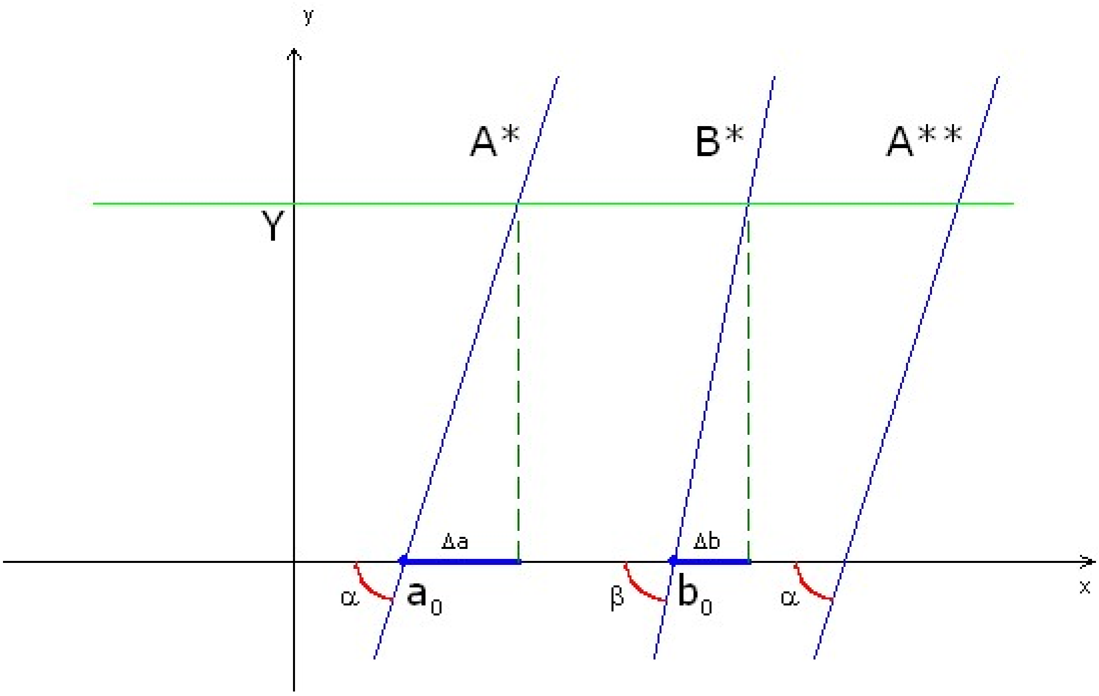
Let {(a,c)} be a point on A* (i.e. {(a,c)}   A*) with (a,c) otherwise unrestrained. We will consider the derivatives da/dc and dc/da relating to the possible movement of the point on A*.

Let {(b,f)} be a point on B* (i.e. {(b,f)}   B*) with (b,f) otherwise unrestrained. We will consider the derivatives db/df and df/db relating to the possible movement of the point on B*.

We must first show that da/dc exists for A and db/df exists for B. Then we must show they are different, i.e. da/dc ≠ db/df. Finally we must conclude that A* and B* meet on the requisite side by examining a line of variable position held parallel to the x-axis. The proof ultimately relies on the notion that a meeting of this line with said two others at a unique singleton is within range if the x-y plane is infinite in extent. Finally this result must be considered in relation to the possibility of changes in the coordinate system context.

At various points in the proof, we will meet the notion of an angle. Immediately after the proof, there appears a reasonable definition of the angle, sufficient to supply some common notions and to underpin the proof. Where reliance is placed on the angle notion, I have inserted the symbol ♣ followed by a superscript as in ♣[35] for example. These superscripts refer to notes in the appendix.

The three lines are shown below together with the y-axis and a line A** parallel to A*. The line A** is not a subject of our initial consideration, but enters the picture later.



Suppose for some singleton in A*, ~ da/dc. Then dc/da = 0 by condition of linearity for A* (see p. 11). Then dc/da and dc/da has constant form by condition of linearity for A*. This means there is no change in c for any change in a as the point {(a,c)} varies in its position of passing on A*. Let {(a,c)} lie at a point distinct from the x-axis. Such a point must exist, since A* is distinct from the x-axis by assumption. Then c ≠ 0. So c ≠ 0 generally whether {(a,c)} remains at this distinct point or varies in position. Therefore the intersection of A* and the x-axis is nullity. This contradicts the requirement that A* should meet the x-axis. The proof is similar for B*, so that we must conclude that da/dc and db/df both exist no matter where the points (a,c) and (b,f) may roam on their respective lines.

Suppose now $da/dc = db/df$. Then the angle which A* makes to the x-axis is the same as the angle which B* makes to the x-axis ♣[1]. Then the interior angles are equal to two right angles ♣[2], contradicting the supposition that the interior angles are less than two right angles.

With reference to the diagram above, the interior angles are α and $(\pi\text{-}\beta)$ ♣[3].
If $\alpha = \beta$ then clearly $\alpha + (\pi\text{-}\beta) = \pi$. We must conclude that $da/dc \neq db/df$ (π is two right angles ♣[4]). Until consideration of the line A**, while our consideration remains with the two lines A* and B*, we will assume that $\alpha + (\pi\text{-}\beta) < \pi$, i.e. the side that corresponds to Euclid's premise is the topside of the x-axis.

With reference to the line {(x,y): y = Y}, Y may be positive or it may be negative. We have caught the line in the diagram when Y is positive. When Y = 0, the line must have distinct intersection points with the two lines A* and B*. These are $\{(a_0,0)\}$ and $\{(b_0,0)\}$. We may regard a_0 and b_0 as constant. In the spirit of a proof by pertinent example and concluding observation, we may leave until the final concluding observation a consideration of the possibility that this is not so. In the meantime, it suits the proof to have A* and B* fixed in relation to the x and y axes.

Let the line {(x,y): y = Y} intersect the lines A* and B* at the points $\{(a_0+\Delta a,Y)\}$ and $\{(b_0+\Delta b,Y)\}$ respectively. The assertion that A* and B* have a point of meeting then may be taken as an assertion that there is a possible value for Y such that $a_0+\Delta a = b_0+\Delta b$. This assertion can be proved with reference to fundamental notions. The following steps of reasoning do depend of course on the assumption that da/dc has constant form.

$$\Delta a = \int_{c=0}^{c=Y} \frac{da}{dc}\, dc$$

$$\Delta a = (k_a c + \mu)\Big|_{c=0}^{c=Y}$$

$$\Delta a = k_a Y$$

We may employ the same notions to derive $\Delta b = k_b Y$. That is to take k_a and k_b as the values of da/dc and db/df respectively. Substituting for Δa and Δb in the condition of meeting we obtain:

$a_0 + k_a Y = b_0 + k_b Y$.

Clearly this has a solution for some $Y \in R$ given $k_a \neq k_b$ and $a_0 \neq b_0$

Momentarily skipping defense of the three assertions about Δa above, let us proceed to deduce where the meeting occurs, i.e. whether Y is positive or negative at the point of solution.

We may first observe that if angle α is very small then da/dc is very large. If α changes on a case by case basis, monotonically increasing to π, then da/dc reduces on a similar basis ♣[5]. Angle α cannot touch π, but as it approaches, da/dc tends to negative infinity, having passed through 0 at the case where $\alpha = \pi/2$ ♣[6].

A similar observation applies with respect to angle β and derivative db/df.

Whereas the diagram has been drawn with $b_0 > a_0$, and whereas $Y = (b_0 - a_0)/(k_a - k_b)$ it follows that for the diagram, Y will have the same sign as $k_a - k_b$. With reference to the assumptions, k_a = cotangent(α) and k_b = cotangent(β), so this sign is positive if $\beta > \alpha$ and negative if $\alpha > \beta$, but $\beta > \alpha$ must apply in order to make the sum of the interior angles less than π (because $\alpha + (\pi-\beta) < \pi$ by assumption).

Now let us refocus on the lines B* and A**.

If contrary to the diagram $b_0 < a_0$, then Y will have the opposite sign to $k_a - k_b$. In comparing this case, we may observe that the lines B* and A** make a pair of lines being fallen on by the x-axis, as required by the conditions of the fifth axiom and that (by design) $k_a - k_b$ still applies as the difference of derivatives, A** being parallel to A* ♣[7]. In this case, the interior angles fall on the other side of the x-axis, indeed on the underside, but they are still α for A**, and $\pi-\beta$ for B*, the same as before. $\beta > \alpha$ must still apply in order to make the sum of those interior angles less than π.

In answer to the question "What if $\beta < \alpha$?", one may answer that in labeling the lines, A* and B*, we may choose such that $\beta > \alpha$.

Let us now turn to defense of the three assertions at the bottom of the preceding page. Here $k_a c + \mu$ is an anti-derivative proceeding out of an evaluation of the integral. μ is a real length and μ has constant form.

The first equation may arise from the definition of the meaning of the integration sign, as outlined in *Russell's Paradox -Two Sets*, (refer pp. 101-105) although in this instance the limits defining beginning and ending of integration path have been filled in from the context. Implicit in the definition is the assertion that if there exists a function whose rate of change with respect to c is equal to da/dc throughout the integration path, then the integral's value can be calculated with reference to it. Indeed it will suffer the same change over the path of integration as will a. Since da/dc constant throughout the integration path, (value assumed is k_a) we may use $k_a c + \mu$ as the reference function provided μ has constant form.

In a standard calculus text one may be treated to a definition of the integral which first contemplates integration of some general function of the variable of integration and does not immediately introduce any derivative symbol. Later when the derivative symbol enters, we may be treated to the Fundamental Theorem. Do we need the Fundamental Theorem of integral calculus to deduce things about anti-derivatives? Not necessarily.

The approach outlined in *Russell's Paradox - Two Sets* more or less casts the integration symbol as designating change in mother function rather than designating limit of a sequence. To be more exact we may consider the following theorem proposed and ask our questions of it. Following the style of *Russell's Paradox - Two Sets*, the (def) symbol after the umbrella of generalisation indicates that the umbrella shuts out indefinite factors.

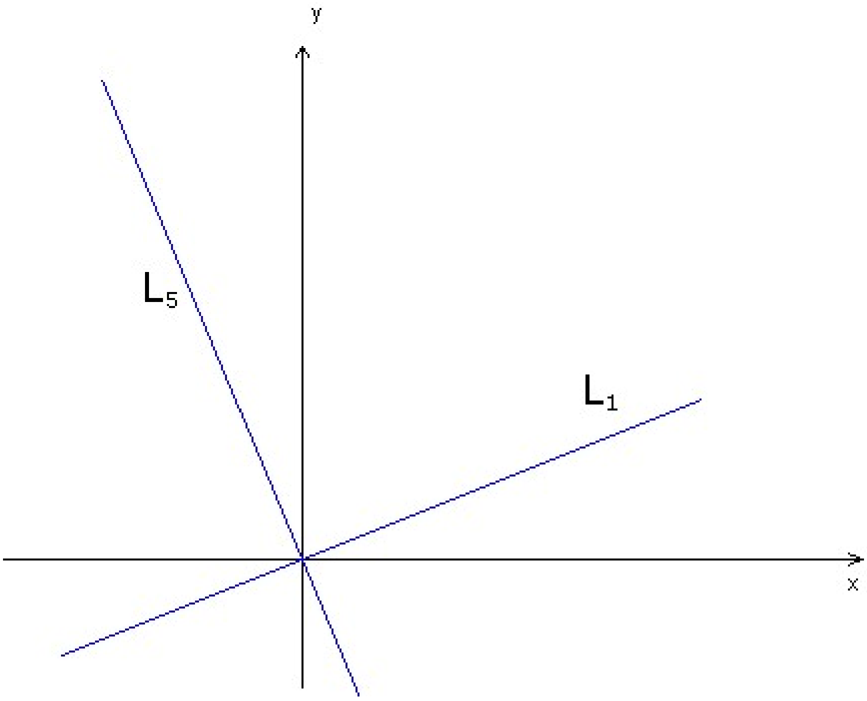
(See pp. 67 - 69 of *Russell's Paradox - Two Sets*. Also on p. 46, indefinite factors are carefully distinguished from identities with names beginning with the indefinite article. An example of the latter type of name is *a number less than 5*. While these types are quite generally excluded from the compass of the free variables in mathematical theorems, it is yet possible and useful to include another type. An indefinite factor has a name which typically begins with the phrase *one of* and follows with a classname, e.g. *one of the numbers less than 5*. Theorems can shut out these types by use of a qualifier such as (def). The qualifier is relevant in the following because variables are not bound merely by an integration sign. Readers may note also that the integration sign in what follows does not signify an indefinite integral but rather an integral with bounds understood. They have been left out. The theorem asserts something applicable whatever the integration path may be.)

$$\text{In general (def)} \; \exists \int \frac{da}{dc} dc \;\Rightarrow\; \int \frac{da}{dc} dc = \Delta a$$

Whether through the Fundamental Theorem or through the definition of integration, we should find that this theorem holds. If we are setting aside the Fundamental Theorem as a little ho-hum, then the theorem shown above can be taken as codifying what we mean by saying that the integration sign is an infinitesimal's redeemer. We can take the anti-derivative primarily as result of an anti-differentiation process being performed on a derivative. The process seeks a suitable reference function for the evaluation of an integral.

Now let us acquaint with a definition for *angle* that supplies the background notions we require for the proof. The graph in the appendix may help some readers follow the definition's development.

The Definition of the Angle



Here we see two lines L_1 and L_5, being infinite straight lines in the sense introduced above and meeting at the origin. By construction L_1 and L_5 are related such that dy/dx for L_1 is the minus reciprocal of dy/dx for L_5. Because of this, one of the two derivatives must lie between 1 and -1, or else one of them is equal to 1 and the other one equal to -1. In framing the diagram, I have chosen so that dy/dx for L_1 is between 1 and -1. I have assumed it is greater than 0. Let its value be k_1 and let the value of the other derivative be k_5.

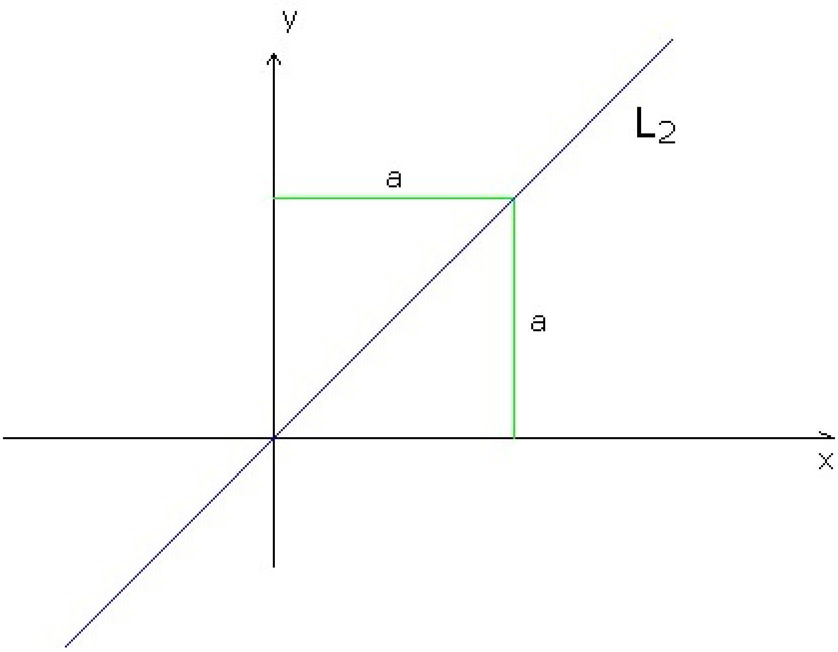
(In using expressions like "dy/dx for L_1", I follow a shorthand in which the symbol for a relation is used also as an elliptical name for the corresponding geometrical, set-theoretic object. Such a name works in a limited set of contexts.)

We may observe that k_1 lies within the domain of the inverse tangent function assuming the latter to be defined by its well known series expansion.

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

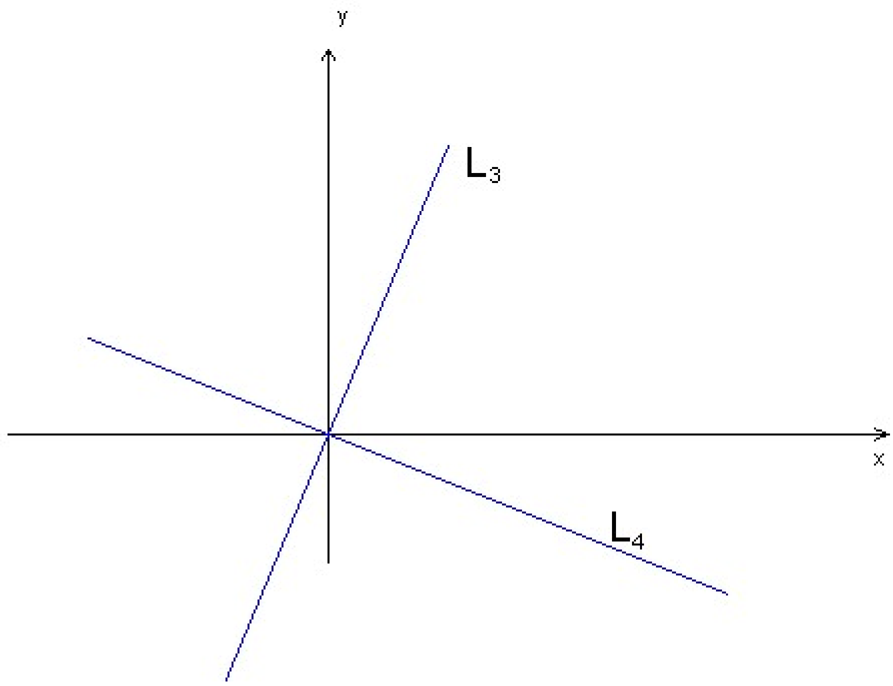
This series converges for $-1 \le x \le 1$. It gives us an angle as a limit for a sequence of partial sums of numbers. Expanding this idea, we may define the base angle of a line by employing a simple formula involving the inverse tangent function. The angle between two infinite straight lines then arises in reference to this base angle. While the picture for *angle* has further complications if we are interested in angle contexts other than Euclid's axioms, it appears that the angle between two infinite straight lines can serve as a useful starting point.

In the first step, the base angle of an infinite straight line is defined in relation to dy/dx for the line. There are five cases, with the first being the case in which $0 \leq dy/dx \leq 1$. As suggested above, in this case the base angle for the line is the exact value of $\tan^{-1}(dy/dx)$. This gives us the base angle for the special subsidiary case in which $dy/dx = 1$. Due to the symmetry exhibited in the following diagram, we now wish to define the base angle for the y-axis and other lines for which dy/dx does not exist. For these lines the base angle must be twice the base angle for the above-mentioned special subsidiary case. This turns out to be $\pi/2$.



Diagrams along the lines of the above may be considered as illustrating that (i) a straight line is two right-angles and (ii) half a right-angle is an important primitive angle.

The third case then arises as the case in which $dy/dx > 1$. For symmetry reasons again, the base angle in this case is $\pi/2 + [\tan^{-1}(-dx/dy)]$. Notice how the argument of the inverse tangent function has switched to a minus reciprocal. An example of a line answering to this third case is the line L_3 in the diagram below.



In the fourth case, $-1 \le dy/dx \le 0$. An example is L_4 in the diagram above. In this case the base angle is $\pi + \tan^{-1}(dy/dx)$.

In the fifth case, $dy/dx \le -1$. In this case, as in the third case, the base angle is $\pi/2 + [\tan^{-1}(-dx/dy)]$. Only this time round, we expect the result to be greater than $\pi/2$. See L_5 three diagrams up for an example.

Proceeding from the above five cases, we may observe that the base angle increases inversely with dx/dy. When dx/dy is large positive, the base angle is close to 0. As dx/dy diminishes the base angle climbs. (dx/dy is called "slope reciprocal" for the purposes of the diagram in the appendix.)

The Angle Between Two Infinite Straight Lines

The angle between two infinite straight lines is usually indeterminate until context or additional nomenclature is employed. This must be so even if the two lines are completely known and do intersect, because there are two angles answering to the description. In Euclid's fifth axiom, the language employed to distinguish between them involves reference to a side, as in "the interior angles on the same side".

If A* and B* are two infinite straight lines in the sense defined above, then to calculate one of the angles between them, we may take the greater base angle and subtract the smaller base angle. Then the second angle between them may be calculated by subtracting the first angle from π. This ensures that both of the angles are greater than or equal to 0 and also less than or equal to π. It also ensures that the two angles answering to *the angle between A* and B**are the same two angles which answer to *the angle between B* and A**. The order of mentioning the lines in this context is therefore unimportant.

In certain cases, the two angles will be 0 and π. Pairs of lines for which this applies may be called parallel. Strictly speaking such a pair of lines makes for a degenerate case, since it is common practice to refrain from referring to the angle between two parallel lines. The reference to an angle between is usually reserved for the case in which the two lines have one and one only point of meeting. The degenerate case may be included in the definition, but talk of sides will enable us to distinguish between the two angles only in the non-degenerate case. In this latter case there is a unique point of meeting and we may focus on that point to decide how the context or extra nomenclature selects one of the two angles. Because of the role played by π in this case, it easily emerges that supplementary angles on a straight line add to π and from this that vertically opposite angles are equal.

Constant Form and Linearity

On http://www.skybicycle.biz/numtheory.htm the reader may download an extract from *Russell's Paradox - Two Sets* wherein I outline a proof of the so-called divide-by-nine property of the natural numbers. It concerns how we get the same remainder, after the dividing by nine of any natural number, as after the dividing by nine of the sum of digits in the number. The extract is taken from the middle of the book, after some preliminary argument about the need to distinguish between classes and sets. In rejecting the Ernst Zermelo solution to Bertrand Russell's paradox, I argue that mathematics must acknowledge a logic of class nomenclature that is quite different from the logic of algebra pertaining to set theory. We really cannot get away without using classnames like *the possible values of a natural number* and we cannot get away from the logic of their formal construction.

In the construction of classnames we can distinguish between values and variables. Indeed the notion of constant form for a derivative involves an idea that there exists a value. Strictly speaking however it is not that the derivative is equal to the value throughout the logical universe. Rather there is a domain of form. Propositions which describe the domain of form are conjoined and the derivative is equal to the posited value wherever these antecedents are true. As briefly indicated in the above-mentioned extract, values are oftentimes distinguished by the circumstance that their names are generated typographically.

With reference to the proof of Euclid's fifth axiom given above, what would be the derivative dc/da at a point (a,c) not on the line A* or A**? Because of the constraints we take on board for (a,c), it must follow that dc/da does not exist at such a point. Therefore the constancy of form for dc/da cannot be a case of its being equal to some value throughout the logical universe. The constraints which describe the domain of form will exclude most of the logical universe.

A typical introduction to calculus in the mold of the Zermelo canon is given in *Calculus - New Zealand Mathematical Society Undergraduate Textbook ISBN - 9597579-5-3*. The authors follow the authors of many similar texts from other nations in developing the derivative first as a function $f'(x)$ related to an original function $f(x)$. Then when introducing the Leibniz notation, they seem to ignore that dy/dx does not mention the function argument. There is no formal identification of the point where the derivative is to be considered. Consequently there is no formal way to set out the above-mentioned constraints and the cross-over between the functional notation of the Zermelo-Fraenkel set theory and the Leibniz derivative is left somewhat loose.

Once we bring class nomenclature into the language of mathematics, we can define the fundamental limit in such a way that its point of evaluation is left silent. We can adopt a variable set, a singleton, that is quite generally where the limit is to be considered. Constraints describing the domain of form for a derivative may then include propositions which refer to this singleton, typically constraining it to be a subset of some larger set. Details are provided in *Russell's Paradox - Two Sets*. The details may help some readers understand how this approach can be sufficiently general.

In the following theorem, we assume a, b are real numbers and CP is the common constraint join wherein the notion of the real number is given substance in relation to a class of real values. (A partial introduction to the common constraint join appears in the above-mentioned extract. The book contains a fuller introduction.) TLS is the above-mentioned singleton. Material implication is denoted by the single-line arrow $\rightarrow$ and logical implication is denoted by the double-line arrow . The meaning of logical implication is that the class of places where the antecedent is true is included in the class of places where the consequent is true. The meaning of material implication is given by the familiar truth table. The first, primary material implication indicates a truth function about the structure of the logical universe.

In general (def) {(a,b): K} is linear without endpoints $\rightarrow$

CP and TLS $\supset$ {(a,b): K} $\Rightarrow$

(1) $\exists \frac{da}{db} \rightarrow \frac{da}{db}$ has constant form

(2) $\sim \exists \frac{da}{db} \rightarrow \frac{db}{da} = 0$

(3) $\exists \frac{db}{da} \rightarrow \frac{db}{da}$ has constant form

(4) $\sim \exists \frac{db}{da} \rightarrow \frac{da}{db} = 0$

In reading the above theorem, how do we know that (a,b) is on the line {(a,b): K} and that (a,b) is not a fixed point? Strictly speaking, I have left these premises out. They ought to be conjoined in the antecedent of the logical implication for a completely formal account. However in the context, the deployment of the variables a, b as bound in {(a,b): K} (as opposed for example to {(x,y): K}) may serve as an indicator that these two extra premises are to be understood.

If the Leibniz derivative is defined as outlined in *Russell's Paradox - Two Sets* then a plurality of parallel lines will also satisfy the above definition as will a line segment so defined as to exclude the end-points. I have omitted giving a full definition for a single, infinitely extending straight line because the proof assumes that the sets A* and B* are of this type. The salient properties required by the proof are those given in the above definition.

Change of Coordinate System Context

Set-theoretic names did not exist for lines in Euclid's time and the geometry which concerns us must be relevant to practical applications. This means that the lines we are concerned with may be defined otherwise than through their set-theoretic names.

While the base angle of an infinite straight line will change after rotation of the coordinate system about the primitive axis, the angle between two such lines, as defined above, will not change. This result applies since the latter is based on the difference between two base angles. Let us confirm this by considering a couple of examples, focussing on the lines A* and B* of the proof and assuming that α (starting base angle of A*) is less than β (starting base angle of B*), as in the diagram on p. 4. As we rotate the coordinate system, the x-axis will separate out from the line that falls on A* and B*.

In the first instance let us rotate the coordinate system through a positive angle γ so that the base angle of B* becomes $\beta + \gamma - \pi$ while the base angle of A* becomes $\alpha + \gamma$. Before this rotation the two angles referred to with *angle between A* and B** are $\beta - \alpha$ and $\pi - (\beta - \alpha)$. After the rotation, the two angles are $\alpha + \gamma - (\beta + \gamma - \pi)$ and $\pi - (\alpha + \gamma - (\beta + \gamma - \pi))$. A little algebra will drop out the gammas and yield $\pi - (\beta - \alpha)$ and $\beta - \alpha$ respectively. These are the same two angles we started with.

In the second instance let us rotate the coordinate system through a negative angle γ so that the base angle of A* becomes $\alpha + \gamma + \pi$ while the base angle of B* becomes $\beta + \gamma$. After this rotation, the two angles referred to with *angle between A* and B** are $\alpha + \gamma + \pi - (\beta + \gamma)$ and $\pi - (\alpha + \gamma + \pi - (\beta + \gamma))$. A little algebra will drop out the gammas and yield $\pi - (\beta - \alpha)$ and $\beta - \alpha$ respectively. Again these are the same two angles we started with.

The examples above do not exhaustively cover the territory of course, but in general after a coordinate system rotation, we will either be adding the same multiple of π to both base angles or the multiple of π added to one of them will be one less than the multiple added to the other one, as in the examples. A coordinate system rotation may change which of the two base angles is the larger, but it will not change the two possibilities which arise for the angle between the lines after one computes the difference between base angles. After such a rotation, the line which falls on the two others for the purposes of the problem set-up will likely not be the x-axis, but it will still have the same angle relationships with the two lines which it falls on. Can there by any change to the fact of a meeting point for these other two lines? To answer this question, let us reflect that the lines are primarily classes of place. As such, properties may belong to them independently.

If the lines are static during the coordinate system rotation, we should expect any point in common to remain a point in common. Let us say the same for a coordinate system translation, i.e. for any change which involves a coordinate system translation, if the lines are static then a point in common will remain in common. One might argue about the meaning of *static* here. However a little consideration may also lead one to the following policy: a change is not valid unless it preserves any facts of geometry which occur whether or not there are coordinates. So from the beginning if we are trying to find some changes which preserve the facts of geometry, changes involving rotation and translation of the coordinate system should appear to be fine, as long as the lines are understood to exist primarily as classes of place in the logical universe.

These expectations may lead us to some transformation equations, allowing us to calculate the new names from the old names and the parameters of the change. In so far as common points are represented by solutions to equations in the names, we should find that solutions do not appear or disappear by the changing of the names.

Turning to the question of the spacing between the lines which are "fallen upon", we may note that the proof does not depend upon the spacing having any particular value, but only upon the spacing being finite in extent where the lines are "fallen upon". Therefore the proof should hold even if the spacing is in a state of change.

References:

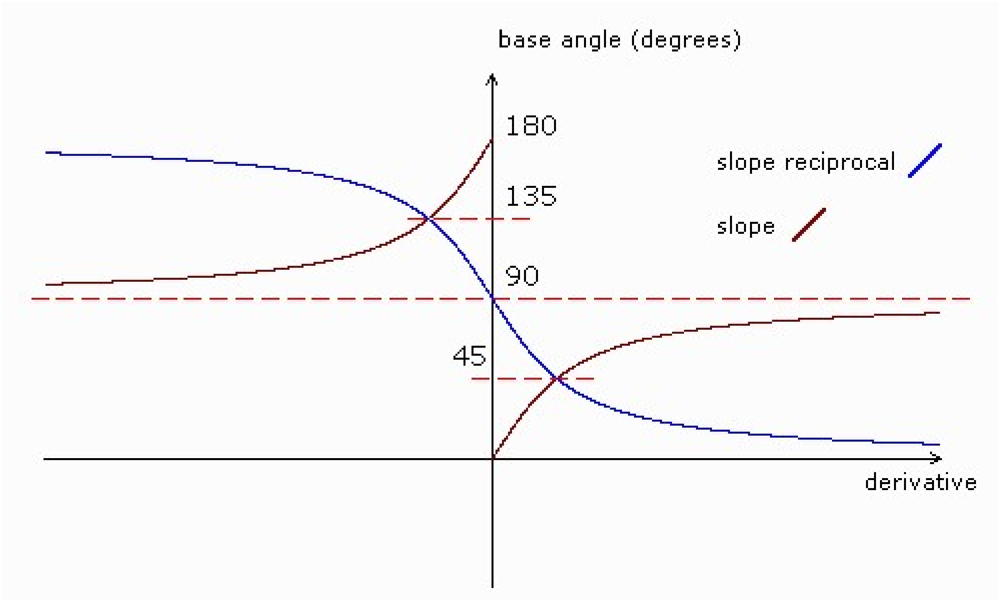
1. An excellent introductory account of the intimacy between the theorem of pythagorus and Euclid's 5th axiom is given on http://mathworld.wolfram.com Some proofs of the theorem of are discussed. According to information provided on the website, *MathWorld* has been assembled over more than a decade by Eric W. Weisstein with assistance from thousands of contributors.

2. *Russell's Paradox - Two Sets* by Russell Z Christensen, Morepork Mystics and Research Team Ltd, copyright 2012 ISBN 978-0-473-20648-2

3. *A History of Mathematics (2nd Edition)* by Carl Boyer, as revised by Uta Merzbach, copyright 1991 John Wiley and Sons ISBN 0-471-54397-7

Appendix

The rules defining the base angle for an infinite linear line in two dimensional space yield the following graph:



With reference to the above graph, the following points are given in amplification of the reliance placed on the angle concept, in the proof:

1. da/dc and db/df are the slope reciprocals for the lines A* and B*. As is evident in the graph, when the slope reciprocals are equal, the base angles are equal, i.e. the angles which A* and B* make to the x-axis are equal.

2. The definition of angle provides that there are two angles between any two infinite straight lines meeting at a point. The definition specifies that their sum is 180 degrees or π radians. If two lines like A* and B* are parallel and cut the x-axis, then their base angles are equal and therefore the interior angles they define on the x-axis must also add to 180 degrees or π radians. This follows from the designation of the interior angles.

3. Designation of interior angles. See also the above points.

4. That $\pi/4$ is half a right angle follows from the symmetry that is implicit in the term 'right angle', together with the angle definition. It follows that π is two right angles.

5. k_a and k_b are the constant values of da/dc and db/df respectively. As shown in the graph, these slope reciprocals vary motonically and inversely in relation to the corresponding base angles. This is observed in

preparation for ascribing a meaning to the sign of $k_a - k_b$.

6. See the graph above.

7. The crux of the proof is that $a_0 + k_a Y = b_0 + k_b Y$, where a_0 and b_0 are the x-intercepts for the lines A* and B* respectively and where Y is the y-value of the point of intersection supposed. A similar result must apply for the lines A** and B*. But the slope reciprocals for A* and A** are equal (they are parallel lines and the slope reciprocals exist) therefore the sign of $k_a - k_b$ does not change when we take k_a to be the constant value of the slope reciprocal for A** rather than for A*.

ISBN 978-0-473-31672-3