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The Divide-By-Nine Property

The proof by pertinent example and concluding observation is something closely related to reductio-ad-absurdum. However its validity may rest on provisioning the logical universe in a certain way. A certain proof by pertinent example and concluding observation poses a challenge to our expression of the principle of induction in number theory. By examining a proof that every natural number has the divide-by-nine property, we can see them both at work.

In what follows I use the term *divide-by-nine property* to refer to a specific property possible for natural numbers. Most likely this coinage has no widespread relevance, but it seems to suit the current text well enough.

It would seem an odd coincidence that upon adding up all the digits of a number's decimal representation some useful information about the number is preserved in the result. Indeed if one is to do this for a number evenly divisible by 9, the result will also be evenly divisible by 9. It is worth trying a few examples to see this. One may also see the corollary that if there is a remainder after dividing by 9, the remainder will be the same if the operation is performed on the sum of digits, instead of the number itself. However failing to see a counter example is enough only for the truly sagacious. Some of us should like to go further and prove that the result must hold throughout the realm of natural numbers.

Examples:
For 7088 the sum of digits is 7 + 0 + 8 + 8 = 23. After dividing 9 into 23, we get the remainder 5 (23 = 2x9 + 5). Similarly after dividing 7088 by 9, we get the remainder 5 (7088 = 787x9 + 5).
For 81 the sum of digits is 8 + 1 = 9. Both 81 and this sum are evenly divisible by 9.

Let x be an unconstrained natural number. Here is our pertinent example - a variable fixed by its constitution to be one of the natural numbers, but permitted, within the logical universe, to have every form of variation consistent with natural number status. Let us consider a showing that *x has the divide-by-nine property* Þ *x+1 has the divide-by-nine property* , let us then whip up an intermediate theorem from the showing. Finally let us see what the principle of induction can make of the intermediate theorem in order to provide the result we desire.

In progressing from the case of x to the case of x + 1, the possibilities in value progression can be divided into two cases which together cover all the possibilities.

<u>Showing That Divide-by-nine(x) Þ Divide-by-nine(x+1)</u>

Progression Type 1: The progression (from the case of x to the case of x + 1) raises the last digit of the value by 1 without changing the other digits. (e.g. 456 to 457, 88 to 89). In this case, the sum of digits is raised by 1; the sum of digits suffers the same increment as the value itself.

Progression Type 2: The progression (from the case of x to the case of x + 1) changes the last digit of the value from 9 to 0. In this case the progression either raises one digit somewhere else in the value by an increment of 1 or it appends a new 1 at the front. Along with this, more 9s may change to 0s. (e.g. 79 to 80, 34999 to 35000, 99 to 100).

In type 1, because the sum of digits suffers the same increment as the value itself, the remainder after division by 9, if it was the same before the progression, as between the value and the sum of digits, must be the same after the progression. Turning to type 2, we may note that any change of 9 to 0 will not alter the remainder after dividing the sum of digits by 9. So the progression will cause this remainder to increase by 1 or to change from 8 to 0. But in division of the value itself by 9, the progression will also cause the remainder to increase by 1 or to change from 8 to 0.

As the two progression types mentioned above cover all the possibilities for the progression of x to x + 1, we may infer that if x has the divide-by-nine property then so does x + 1.

The astute reader, in reflecting on the showing given above, may decide that there are some hidden assumptions. But let us gloss over them for the moment and proceed to the next step of the proof. The next step of the proof is a declaration to the effect that if x's relevant properties are made antecedent, the result is a theorem generalising to all possible constitutions for a natural number, indeed to encompass all of the numbers generated by the successor function, as well as natural number variables.

But exactly what form will the intermediate theorem have? Here is one possible candidate: (Let us assume that *Nat* is short for *the set of natural numbers*.)

In general x ˛ Nat fi [divide-by-nine(x) fi divide-by-nine(x+1)]

What about the following theorem instead?

In general x Û 1(M) fi (CP Þ [divide-by-nine(x) fi divide-by-nine(x+1)])

Here there is first mentioned a property of belonging to a class of possible identities. M figures as the possible identities of a natural number. Then mentioned next, there is a conjunction of propositions we may symbolise with CP – the common constraint join. Finally if the two antecedents are true, the theorem holds us to find that the divide-by-nine property transfers in general from the case of x to the case of x + 1.

In what follows, this second version of the intermediate theorem is called the values opportunity theorem.

An Inference into the Qualified Universal Quantifier

Out of the intermediate theorem, pursuant to proving by induction that a certain universality belongs to the divide by nine property, we may seek to infer the following local proposition. We are prepared to grant that truth adheres to CP.

"x([divide-by-nine(x) ﬁ divide-by-nine(x+1)])

However the universality given by the umbrella clause “In general …” is qualified in the values opportunity theorem with the antecedent

x Û 1(M)
x is one of the possible identities of a natural number

Therefore the universality of the values opportunity theorem does not licence the unqualified universal quantifier, in the expression:

"x([divide-by-nine(x) ﬁ divide-by-nine(x+1)])

Rather it licences only a reduced scope of universality. In suggesting that we draw off the following proposition, as a consequence of our acceptance of CP, I am not suggesting that here is the definitive and unique, the one-and-only local inference. It is rather an inference that fits.

"x ˛ Nat([divide-by-nine(x) ﬁ divide-by-nine(x+1)])

At every location in the logical universe where CP is true, the qualified universal quantifier in the above proposition has the same meaning. Well, almost. Certain identities may appear and disappear from the scope of the universal quantifier, in stepping from one place in the universe to another. That is because the existence behind the meaning of the quantifier (i.e. membership of the class of mathematical objects) is a local property. But the values exist in this sense everywhere. Which values are invoked by the quantifying clause: "x ˛ Nat? At least that much remains the same from one location to another.

This is interesting as the principle of induction may admit of being put as follows, at least for the case of its application to the divide-by-nine property:

In general
[divide-by-nine(0) and "x ˛ Nat([divide-by-nine(x) fi divide-by-nine(x+1)])] Þ
"x ˛ Nat(divide-by-nine(x)).

This theorem would follow Howson, p. 20, item (iv), closely enough to be thought roughly equivalent to Howson's formulation for the case of the divide-by-nine property. If we put it into the context of the values opportunity theorem, we should see it providing a basis for an inference to the following theorem:

In general CP Þ "x ˛ Nat(divide-by-nine(x)).

Broken into simple steps of reasoning, the inference runs as follows, starting with the intermediate values opportunity theorem.

In general x Û 1(M) fi (CP Þ [divide-by-nine(x) fi divide-by-nine(x+1)])
∴
In general CP Þ "x ˛ Nat([divide-by-nine(x) fi divide-by-nine(x+1)])
        (interpreting the meaning of M and CP)
In general CP Þ divide-by-nine(0)
        (adding a prior result assumed from number theory)
∴
In general CP Þ
    divide-by-nine(0) and "x ˛ Nat([divide-by-nine(x) fi divide-by-nine(x+1)])
        (conjoining the consequents)
In general CP Þ "x ˛ Nat(divide-by-nine(x))
        (by induction)

What about divide-by-nine(0)? Let us turn to the Peano axioms. While these are normally given in local propositions, e.g. as in Howson on p.20, we may now wish to write them as follows:

(1) In general CP Þ 0 is one of the natural numbers.
(2) In general CP Þ
    [x is one of the natural numbers fi x+1 is one of the natural numbers]
(3) In general CP Þ
    [x is one of the natural numbers fi x+1 „ 0]
(4) In general CP Þ
    [[x is one of the natural numbers] and
    [y is one of the natural numbers] and
    [x+1 = y+1]] fi
    x = y
(5) In general [p(0) and "x ˛ Nat([p(x) fi p(x+1)])] Þ "x ˛ Nat(p(x)).

Do these axioms provide divide-by-nine(0) as an inference? They provide CP Þ divide-by-nine(0), when assisted by some other foundation theorems. As we may take for granted the truth of CP, so we may take for granted divide-by-nine(0). However there is an implicit connection between the symbols Nat and CP.

The Class of Possible Values

To be sure, I am not sure: what device for the expression of constancy is available, for one who religiously follows the Zermelo-Fraenkel formulae precedent to the Howson book. However the word *value* is deeply buried in common usage, in mathematics lectures and in various other contexts where numeracy is taken seriously.

Building it in, the implicit connection between Nat and CP can be summarised as follows: CP - at least in part - represents propositions which give meaning to the defining property for Nat. What does it mean to be equal to one of the possible values of a natural number? CP stacks out the class.

Indeed the following proposition may occur as a conjunct of CP: *0 and the numbers generated by the successor function are the possible values of a natural number.*

A Chance that CP might Not be The Case

Now if we should wish to employ the symbols 0, 1, 2, 3 in a modulo-four context - for example supposing we would wish to have 3 + 1 = 0 - then we could deny CP, in effect to deny that 0 and the numbers generated by the successor function do function as the possible values of a natural number. We would have some other name stem to apply to our theoretical object, should 3 + 1 = 0 be true in respect of its progressions.

In contemplating the function of a conjunct such as *0 and the numbers generated by the successor function are the possible values of a natural number* we may reflect upon the making of a reference to an essentially mechanistic or typographical rule. In the example, such a reference is understood in the term *successor function*. The successor function contains the general rule of generation for what are value names as opposed to value descriptors. Such a conjunct plays the role of introducing the class of possible values of a theoretical object by means of a typecasting for value names. Mere value descriptors are not required to keep to the typecasting, but the descriptors are in a sense only secondary names.

(section omitted in PDF extract for contextual reasons)

A Flagship Inference

It is an elementary exercise to deduce from the Peano axioms, as stated above, the following theorem:

In general CP Þ "x ˛ Nat(x is one of the natural numbers)

This inference is possibly a flagship inference for induction. It cannot be taken completely for granted if the defining property of Nat is the property of being equal to one of the possible values of a natural number.

Following the example outlined above relating to the divide-by-nine property, the critical step involves moving from axiom 2, viz:

In general CP $\Rightarrow$
[x is one of the natural numbers $\Rightarrow$ x+1 is one of the natural numbers]

to the following theorem:

In general CP $\Rightarrow$
$\forall x \in$ Nat[x is one of the natural numbers $\Rightarrow$ x+1 is one of the natural numbers]

The step interprets the meaning of the umbrella of generality. It is intended that the free variable unrestrained has a greater scope of possible variation than any bound variable, so in binding the variable to the universal quantifier we express a summary of some of the substitution instances applicable to the variable.

The flagship inference expresses something which is not necessarily obvious. Indeed if $x \in$ Nat then it still may be the case that $\sim\exists x$ at some locality in the logical universe. At such a place, it certainly would not be true that x is one of the natural numbers. However the universal quantifier in the clause $\forall x \in Nat$ blocks out any identity which has locally fallen into non-existence. It does not speak for such an identity, because even before we consider the meaning of the set membership symbol, the meaning of the universal quantifier intrudes. Based on theorem **(I)** the universal quantifier has a local meaning, generalising only to cover the mathematical objects that locally exist.

Proceeding to follow the flagship inference, after the critical step outlined above, we are called to apply the theorem of induction. The formulation given above for induction does however gloss over a certain problem of scope for p(x). Let us consider that formulation, repeated below for convenience, more closely.

In general [p(0) and $\forall x \in$ Nat([p(x) $\Rightarrow$ p(x+1)])] $\Rightarrow$ $\forall x \in$ Nat(p(x)).

The proposition of logical implication which appears here under the umbrella of generality is true only if there is somewhere in the logical universe where its premises are true together. There will be some substitution instances for p(x) which will fail on this count. Consider however the following formulation:

THEOREM OF INDUCTION

In general
CP fi [p(0) and "x˛Nat([p(x) fi p(x+1)])] Þ
CP fi "x˛Nat(p(x)) …**(XX)**

Assuming there is somewhere in the logical universe where CP is false, then the proposition of logical implication in this formulation is always afforded a place of truth for its antecedent. Of course if that place of truth is a place where CP is false, then the consequent supplied by the logical implication of induction will be of no practical use.

In the flagship inference, in what may be considered typical form, we have the following two premises to present to induction:

In general CP Þ 0 is one of the natural numbers.
In general CP Þ
"x˛Nat[x is one of the natural numbers fi x+1 is one of the natural numbers]

Treating *p(x)* as *x is one of the natural numbers*, from these premises we may as easily draw off
CP Þ CP fi [p(0) and "x˛Nat([p(x) fi p(x+1)])]
as
CP Þ [p(0) and "x˛Nat([p(x) fi p(x+1)])].
In fact the two propositions are materially equivalent, even under the umbrella of generality.

Following the austere induction, we then conclude
CP Þ CP fi "x˛Nat(p(x))

But once again, even under the umbrella of generality, this is materially equivalent to
CP Þ "x˛Nat(p(x))

In fact, if Q is any formal proposition whatever, then CP Þ [CP fi Q] is materially equivalent to CP Þ Q. Therefore in a typical context where induction is applied, the *CP* fi which rescues the theorem of induction is more or less redundant. It is needed only because the theorem endeavours to cover the untypical context as well as the typical context.

Reserving the Value Sets associated with Special Stems

A purist may fail to be sure that CP will be true wherever a natural number identity finds itself in the class of mathematical objects, but we may as well assume that CP is then true because we may assume the values generated by the successor function are an illustrative sequence.

For the purposes of general mathematics, we may limit our name stems like *natural number* which occur in CP, through an assignment of possible values, so that they do not appear in any other parallel or similar assignment. This would seem a fair condition for the logical universe, from the nature of the assignment, compared with an ordinary contingent proposition of fact. What makes CP contingent is then just the potential for the values like 0, 1, 2 ... to be given various different roles, various different fact sets by various different theories. CP is made contingent in the mold by the understanding that 0,1,2 ... are not everywhere natural numbers.

We do constitute certain symbols as value names first, and numbers of a certain kind second, if we deploy a certain conjunct as a turf-defining means, as outlined above. However in *0 and the numbers generated by the successor function are the possible values of a natural number* we refer to a natural number as a theoretical object. If we should utilise a variable to denote this theoretical object, we should find the variable is bound by its birth. We can add conditions for the variable, but we must express them in a place covered by the binding, e.g. as in *the possible values of a natural number x: x + 1 < 200*. In this case the class name is not complete without the expression of the condition within it. If however there is no condition expressed within the class name, as in *the possible values of a natural number*, then we should expect the referent to contain a full set of values - sufficient values for a number theory to be fully expressed in their compass. The class referred to is then local in the sense that a number theory is only a theory. The logical universe may be large enough to admit a plurality of theories.

The effect of certain well-known theories is that there are claims over the following name stems: *natural number*, *whole number*, *integer*, *rational number*, *real number*, *complex number*. In contriving a new theory, one would hardly wish to tread on this ground. A new and different name stem would be required. However the new theoretical object could then lay claim to a territory in the logical universe. In its territory, symbols like 0, 1, 2 … might be drawn into a foreign equality fact set.

In general we may characterise a number theory as a theory about the behaviour of a theoretical object given a class of possible values for the object. The objects which are introduced as possible values are values first and numbers second.

Of Values and Constants

Let us now consider the term *changing value*.

We should have no problem with the sentence. *The value is changing*. In suitable context, the subject of the sentence is the value of some mathematical object. Such an object may change its value unless it is a value itself.

However there arises a small possibility for confusion, and to draw a line, strictly, whence to reduce confusion, the term *changing value* is to be regarded as something of a contradiction in terms. If it comes to enumerating the possible identities of a value, such nomenclature as *the changing value* is ruled out. If one feels the need of the term, one should try harder to bring into the picture the non-value object for which value is changing.

Further a name such as *the value of x* is also frequently insufficient to identify a value. Consider for example, *the value of the number of walnuts on the tree.* In this name the term *value* is entirely redundant. Also it turns out, in practice, when we are asked to identify a value, if we do not know a name like 0,1,2 – i.e. a name that has been given a value role – then we usually say the value is unknown or we give it a descriptor like x_0.

These considerations underscore the idea that 0,1,2 ... are values throughout the logical universe. What varies from theory to theory is not that they are values, but that the relations between them may take different forms. Therefore if it comes to writing a list of names for the possible identities of a value, we may write such names as *the value 0*, *the value 1*, *the value 2*, ... although equally we may simply write *0*,*1*,*2* ...