Chapter One:
Our Formal Language: Introductory Observations

Hypothetical space is part of the logical universe. In hypothetical space we entertain postulates, without necessarily requiring the postulates to conform to what exists. This is an essential part of the process of discovering scientific law. But it is also used a lot by firefighters. When firefighters are not fighting real fires, they are often fighting fires in hypothetical space. This is because fires come in various different forms and can be complicated in various different ways. It can help to meet a complication of a fire in hypothetical space before meeting it in real life. This gives the firefighter a chance to work out how to deal with the complication while the pressure can be relieved by muffins and coffee.

Formal language is necessary when fighting fires in hypothetical space. Formal language is therefore a language which addresses subject matter like fire, air and the oxides of uranium, as much as it addresses the square root of four. Such language is formal in part because of what it offers to the adept - a path of response, enabling the drilling down to a precision of meaning by a party of two or more, when at least one of them is adept.

So a discussion opens up, about what is formal in language.

Some people thought that a key question was answered in the 1930s, when a way of solving Russell's Paradox in logic appeared to achieve majority support among the mathematicians of the day. Afterward the New Mathematics movement began to usher that answer into general circulation, at least in my home country of New Zealand. However we have now rejected New Mathematics, at least as a wider community, and the question is open again.

Buried in the underlying logic of common English is a form of expression relating to classes that offers another answer to the question. In taking a role already, in ordinary English, the form of expression to which my vote is given appears in law and legal instruments like patents as well as in other places. There is a certain level of community acceptance already for this form.

Then to begin at an appropriate point of beginning, for a subject so intertwined with society as mathematics, let us begin by examining a case at law. The case to be outlined came before the Environment Court of New Zealand. A local authority wished to pass, as a by-law, words that make an unfathomable demand on the residents of a district. The law appears to indicate that the authority had done no reasoning in hypothetical space, except perhaps in relation to the possibility of future 'legal' proceedings and the need to score pre-emptive strikes.

In the lead-up to the case, a draft district plan containing proposed rules of behaviour for the residents of a district had been presented for public comment. Once the plan would be passed, after a process of consultation, the rules would have the force of law. Confiscation of property and two years in jail could result from breaking the law. The plan divided the district into commercial, residential, rural and industrial zones, as was usual. It declared that any resident intending to carry on an industrial activity in the commercial or residential zones would need to apply to the authority for permission. The authority would be able to determine at its sole discretion whether to allow the activity. The activity would be illegal without the grace of the permission. However the following definition would be used to determine whether an activity was industrial or not:

> *INDUSTRIAL ACTIVITY means any activity which involves every part of a process from the receipt of raw material to the dispatch or use in another process or disposal of any product or waste material, and any intervening storage, use, processing or conversion of the raw material, partly processed matter or product.*

It should be easily discerned that the definition covers the activity of breathing and the activity of operating a domestic dehumidifier. In fact the definition covers just about every human activity under the sun.

The matters of language arising from this case do not all reflect badly on the local authority drafting the rule. Certain fundamental features of their language appear again in the language of science. But in using the language for their purposes, the authority made mistakes of fundamental import. I now wish to examine the mistakes and then later to examine the features which were not in error. Proceeding in this order will facilitate a crossing over into the language of mathematics. For indeed the use of formal language and reasoning in hypothetical space are very important in mathematics.

The authority made two main mistakes, frustrating their own and everyone else's comprehension of their law. Firstly they used the occasion of a definition to widen the scope of a common term, whereas definitions should be used to narrow the scope of common terms. Then having defined an absurd class of activity - a class incontinent in its ability to identify any activity apart from another - they outlawed all activities in the class without drawing any exception to the general rule. An exception drawn is really only part of a rule. But rules about behaviour should be polite; the first aim of law is to get the citizen and the law-maker on the same side.

In searching for the polite, exact expression, one may be ill-served by many of the grammar books. In the line-up of names, the list of name types is often too coarse. To have a line-up suitable for logic, one should identify at least five types: proper names, identity names, class names, factor names and quantity names or determiners. Among these, identity names are ubiquitous both in common English and in the English of the law. An identity name must begin with an article and there are only two articles being the definite and indefinite articles. After the article, an identity name has a name stem containing a substantive and possibly qualifiers. The name stem can function as a basis for forming other name types as well as identity names.

The relevant name stem in the case outlined above was the term *industrial activity*. A short descriptive phrase viz: *industrial* accompanies a noun viz: *activity* to form a name for some doing or thing. From the name stem, we can form a class name *the industrial activities* and from the class name we can form the factor name for a member of the class: *one of the industrial activities*.

It is quite impractical to define every name stem in a tract of law such as a district plan. Of course such a thing must rely heavily on the common language. But occasionally the common language offers only a complex, lengthy construction where the writer wishes to convey a precision of meaning. By the time various qualifiers are compounded with each other, the writer faces an ugly, inelegant passage in their writing. On this occasion, a definition can be used. Out comes the ugly, inelegant passage and to replace it there is inserted a much shorter term such as *the industrial activities* for example. The name stem of the inserted term is then defined at the outset of the writing, so that where names based on the stem are encountered in the writing, readers may know that something finer in meaning than the common idea is intended.

There are other causes for definitions, for example in connection with invented adjectives as outlined briefly in chapter two. But if a class of action is to be both a focus for the law and something very broad and inclusive then usually it would be crafted by compounding categories and sub-categories each with their own articulated meaning.

The reason why definitions are normally used to narrow down the meaning of name stems is that the primary reason for there being any law is the furtherance of community interests. Therefore it is paramount that law should communicate clearly. There is no call for a definition other than a definition which facilitates comprehension. Indeed this also applies to mathematics, even though general truths of mathematics are not rules of behaviour.

The local authority's second main mistake was really an extension of the first, but it occurred in a different place within the district plan. It involved a failure to use the word *except*. Whereas such a word and its leading are frequently very relevant to both laws of behaviour and general truths of mathematics, there probably has not been sufficient educational emphasis on this word. The second main mistake occurred where in the district plan the name stem *industrial activity* was incorporated into a class reference, involving pluralisation of the term.

From *industrial activity*, we obtain the class name *the industrial activities*. Where it came to the point in the plan of specifying a list of Non-Complying Activities, there was writer's opportunity to incorporate a reference to the industrial activities. Proper nouns aside, Item (b) on the list was written as follows:

> *(b) Industrial activities in Sibelius and Waizaki.*

This reference to a class did not involve the definite article *the* and I will comment on that later. For the moment, let us take the wording as equivalent to:

> *(b) The industrial activities in Sibelius and Waizaki.*

However it might well have read:

> *(b) The industrial activities in Sibelius and Waizaki except ...*

where the dots would indicate a reference to some activities considered to be not warranting of the Non-Complying status. The word *except* has a special relevance in relation to plural names.

If a law is to be carefully developed, from reasoning in hypothetical space about the possible ways for life to be lived, we may expect the expression of the law to be complex in nature. One of the easiest ways to incorporate a little complexity is to use the word *except* when coining a plural class name. Then the class to which one refers has both included and excluded aspects.

It was possible to exclude activities by narrowing down the meaning of the name stem *industrial activity* or by using the word *except* in referring to the class of Non-Complying activities. In resolutely refusing both of these options, the authority laid waste to its law-making powers. However in their defence, this error seems to have been authorised from the top in Wellington. Their definition for *industrial activity* was plucked whole from the Resource Management Act, 1991(RMA).

In concluding the proceedings of the case, the judge asked to see an alternative definition for the term *industrial activity*. There was very little that could be done by the appellant at this stage and the rule discussed above was passed into law. Had the proceedings been scripted, the appellant may have been given a better chance. Law without the decision, without the sorting of fear, predicated on a consultation coming after the law rather than before it, can be ill for any cause. Loosely called "the defence of the realm", a certain cause must be held high. It deserves to be heard in learned opposition to all irreverent defining in the law.

In moving on from the authority's mistakes to the graces, two positive features of the language can be seen, tending to assist the power of conveyance in the language.

The first is that plural class names have been employed. I pick on this because they are not given proper place in the formal language of mathematics and this is one way in which mathematics can advance. Plural class names, like names for sets in mathematics, admit of formal construction methods. Words like *except* can enter in to these methods as formal connectives.

The authority's omission of the definite article, in its reference to *industrial* *activities*, rather than to *the industrial activities*, is however justified. In certain contexts, the definite article can be dropped as superfluous. Where it is required, frequently the definite article signals to the reader that context must be fully employed in determining the meaning of a class name. But no context of this type appears in the district plan mentioned above. The class name is used in a list and what comes above and below in the list are unrelated items. So if suitable context were to apply, it would have to occur within the same list item or in the introduction of the list.

Another time when the definite article is formally required is when to omit it would create ambiguity, because of context. But once again, there is no relevant context in the district plan. There being no relevant context, the definite article can be dropped.

But in which way is the authority's law composed? Is the subject matter sets, or is it space? The second positive feature of the authority's language is that it clearly sets forth a law of space. This it does with the phrase *in Sibelius and Waizaki*. As with the first positive feature of using plural class names, I am drawing attention to this feature because it is not currently shared with the way general truths of mathematics are set forth in formal discourse. Yet it is a powerful plank of understanding, joining laws of behaviour with laws of science. The law is about what happens, or should happen, in, around, on, above, etcetera - a space.

Consider the law about the boiling point of water. This law says that where the atmospheric pressure is such-and-such, the boiling point of water is so-and-so. Of course, water does not boil without occupying space, nor do people have industries. While the application of mathematical truth to space may be implicitly understood, yet space itself must be an object of mathematical thought.

It turns out that classes named with plural names alone are not sufficient for mathematics. However mathematics already has another kind of class on board with its set theory. When these two types of class are made complimentary, mathematics can be richer and more encompassing, it can properly aim to write law of space.

If the formal language of mathematics were to be unified with the formal language of law, then the grounding in mathematics, available through the schools, could plant the seeds for a better comprehension of law, a better understanding of how law can be used, like the law of the sea and the law of insurance, to build peace and harmony in the world.

The admission of plural names into formal mathematical discourse is part of the way forward. As important as the plural names themselves will be derivatives of them, such as the factor name for a member of a class. In the example given above, where the class name is *the industrial activities*, the factor name is *one of the industrial activities*. Then by means of mere grammar we have already solved an important mathematical problem in relation to such classes. This problem is associated in mathematics with the axiom of choice.

A layman's introduction to the axiom of choice is given on the PlanetMath website at http://planetmath.org/encyclopedia/AxiomOfChoice.html. To save readers the task of looking this up, I will repeat the story from memory, concerning shoes and socks.

We must imagine that there are infinitely many pairs of shoes and infinitely many pairs of socks. This should be easy enough, if we allow hypothetical shoes and socks to be included. To be concluded by a process of formal logical reasoning, we are to find that there exists a class of shoes in which there is one and only one shoe from each of the pairs of shoes. And then we must do the same for the socks.

In the case of the shoes, there is no problem, because left shoes are distinguishable from right shoes and each pair must contain one of each. So the class of all the left shoes satisfies our search, as does the class of all the right shoes and many other classes whose names can be constructed by mixing lefts and rights. However in the case of the socks a problem arises because left and right socks are indistinguishable. At least for the purposes of the problem, we will assume that the terms *left sock* and *right sock* are informal. The answer to this problem has been thought to require a separate axiom in mathematical set theory, and this has been called the axiom of choice.

Most of the problem is solved when we can name a suitable class, because our natural tendency in seeking a name is to rule out any names of classes which plainly fail to have existing members. Using plural nomenclature then, we may be drawn to consider the following name: *the possible identities of a sock except one of the socks out of every pair*. This is a simple plural name, followed by the connective *except* and then by an almost formal name for a general member: *one of the socks out of every pair*. In fact the only aspect detracting from the formality of the latter factor name is a shortening of form and, as we shall see, this can be allowed into formal discourse as an ellipsis.

The possible identities of a sock is a plural name which may, on first sight, seem to be superfluous. However this name does not invoke context, like the shorter name *the socks*. Rather to the contrary, it allows us to unambiguously refer to the class of all socks, while avoiding any socks which might be logically impossible. As with other plural class names, continuing with the word *except* is natural, if that has not already been done. What we wish to exclude, in this case, from the denotation of the completed name, is one sock from each of the pairs.

Since *every pair* is short for *every one of the pairs of socks*, the only element left to address in the name is the connective element in the words *out of*, since *one of the pairs of socks* is plainly formal. In deploying the definite article, the class name *the pairs of socks* invokes the context of the question framing. *The pairs* then refers us to the pairs of socks hypothetically given. The pairing is a given.

Of course the words *out of* are primitive for a fundamental relation and any course in mathematics, or even in language plain, should teach their meaning.

Having found at least one suitable class name, it would remain to formally prove existence for the class. However we shall discover that existence is really a non-issue for classes: at least there is a productive paradigm which supports this view.

In being a book about a road in formal language, this is also a book about a paradigm for understanding. Do we seek to understand the operation of a pressure cylinder or a nuclear power plant? What then are the basic concepts that we should carry into such an endeavour?

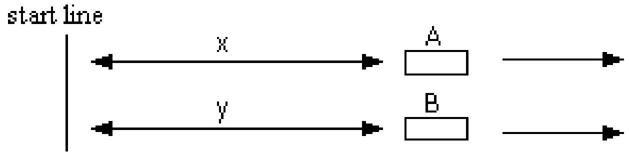
Do these concepts afford us a solution to Russell's Paradox as well as to the dilemma of the axiom of choice? Indeed they may.

Reading through the PlanetMath account of the axiom of choice may give readers an idea that debate about the problem has been intense, particularly in the era when Russell's Paradox was considered unsolved, in the early part of the twentieth century. The history of mathematics can be very interesting when it touches on these debates. For such debates do not permit of outcomes like efforts to find solutions to equations often do. And yet in subtle ways, they can exert influence.

Relevant to the case of the timber treatment plant, or the nuclear power plant, or the effects of Man in general, and close to the heart of the matter of unification, one of the names to note in the history of mathematics is Gottfried Leibniz. We shall see that the paradigm appropriate to the legal mode in technical language is capable of supporting a superior definition of the Leibniz derivative.

Gottfried Leibniz was a German lawyer who introduced the term dy/dx into published mathematics in the year 1684. Although Leibniz was completely self-taught in mathematics and his mathematical investigations were sidelined to his main occupation, his output of mathematics was reasonably prodigious and he has been highly regarded in the history of mathematics. dy/dx means the rate of change of y with respect to x, where y and x are mathematical variables belonging to a common context.

An example may help to make this clear and also to introduce a related problem in the foundations of mathematics. With reference to the diagram below, suppose A and B are two bodies in motion away from a common start line. Let x be the distance of body A from the start line. Let y be the distance of body B from the start line. If B is moving at twice the speed of A then y is increasing by two length units for every one length unit of increase in x. In this case, $dy/dx = 2$, i.e. the rate of change of y with respect to x is 2. At the same time, $dx/dy = ½$, meaning the rate of change of x with respect to y is ½, or that A is moving at half the speed of B.



In this analysis, A and B may have started at different times. But extrapolating, we may hypothesise a general rule to apply to this context, whether or not A and B have started at the same time:

$dy/dx \cdot dx/dy = 1$ (read the dot as a multiplication sign)

In fact, as a general rule, this would not quite be general enough. For consider how the analysis is affected if one of the bodies stops. If A stops while B keeps on going, then $dx/dy = 0$, but we have a problem with dy/dx. In this case, we say that dy/dx does not exist. So to generalise the rule sufficiently to cover the possibility of bodies stopping, we can write:

$dy/dx \cdot dx/dy = 1$ provided both derivatives exist.

The problem does not arise until we seek an equivalent expression for the derivative, so as to render its meaning in terms of the fundamental concept of a limit. It can be useful to have such an equivalent expression, both for calculation purposes and for understanding theories of how things work.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{y\big|_{x=x_0+\Delta x} - y\big|_{x=x_0}}{\Delta x}\right)$$

It turns out that the expression on the right-hand side in the equation above can almost encapsulate what we mean by dy/dx. Moreover the equation above is the sort of equation that we can generalise to a space, so if the law it expresses is fully stated we will see some kind of reference to a space. However the equation is not quite right yet.

When we write a vertical bar after a variable, such as appears after the variable y in the expression above, we make room for two places where propositions can be inserted, one at the top of the bar and one at the bottom. Sometimes only one of the propositions are required. These propositions are said to determine an evaluation context for the variable. So in reading the expression above, we come across "*y evaluated where* $x = x_0 + \Delta x$" and "*y evaluated where* $x = x_0$".

The problem in the foundations arises because equations in the variable x do not always sufficiently determine the evaluation context for y. Consider what happens for example if x = sin(y). Relating this to the bodies in motion, as mentioned above, this might be the case if B keeps on going in one direction, away from the start line, while A goes back and forward in a reciprocating motion. If x = sin(y), then the expression "*y evaluated where x* = x_0" does not have a determinate meaning, since A will cross and recross the point represented by x_0. At each crossing, y will have some new value. Yet dy/dx still means the rate of change of y with respect to x. dy/dx does have a determinate meaning, whereas the limit expression offered above does not, because an equation in x is not sufficient narrowing down of the evaluation context for y.

A natural method of sufficiently narrowing an evaluation context arises in the fullness of the new paradigm. To begin our analysis of the paradigm we will consider Russell's Paradox in logic and survey the three main historically interesting solutions. One of these solutions exhibits an intuition that classes and sets are distinct in the annals of mathematics. It is this intuition which we wish to take further, in the development of the paradigm.

Russell's Paradox

Some sets are members of themselves. For example, consider the set of all sets. On the other hand, many sets are not members of themselves, for example a set of three horses: the set is not itself a horse. So consider the set of all sets which are not members of themselves. If this set is a member of itself, then no, that cannot be. It is defined to include only sets which are NOT members of themselves. But if this set is not a member of itself, then it conforms to the membership criterion and we must conclude that it IS a member of itself. There are no other options. Either it is, or it is not, a member of itself. But both options lead to contradiction. Russell's Paradox suggests that there is an unavoidable contradiction in mathematical logic.

Russell's Solution

After discovering his Paradox in about 1901, Bertrand Russell spent a number of years thinking about it and then offered us a solution. For various reasons, the solution was unacceptable and it is now widely considered to have only an historical interest.

Russell's solution is sometimes referred to as the "no classes" solution. It proposed that a class or a set should be considered to be no more than a function. One reason this solution is so hard to understand is that we may intuitively feel that functions have a certain nature. We may feel that the name for a function is not a complete name until an argument has been mentioned within the name.

An example of a function name wherein argument is mentioned is *the square of a number x*. If we take away the argument name, we are left with *the square*. This is far too elliptical to work as a formal name (unless it means something geometrical), and so we may migrate to a longer name: *the square function*. Let f then be a symbol for a function name, such as is common in algebra and calculus from mid secondary school on. The context might state for example $f(x) = x^2$.

Then what form is implicit for f? Does it mean something like *the square function*, or does it mean something like *the square* - waiting for completion by the addition of argument name. While the question is lame in the example context, because there the argument name does appear, it is not a lame question in the wider mathematics. Indeed Russell felt that function names like *the square function* were formal without the argument being mentioned and that sets and classes could thus be accounted for: they could be considered to be special functions. How does this solve the paradox?

The following comment appears in the Stanford Encyclopaedia of Philosophy on this subject:

> "Russell's own response to the paradox was his aptly named **theory of types.** Recognizing that self-reference lies at the heart of the paradox, Russell's basic idea is that we can avoid commitment to **R** (the set of all sets that are not members of themselves) by arranging all sentences (or, equivalently, all propositional functions) into a hierarchy. The lowest level of this hierarchy will consist of sentences about individuals. The next lowest level will consist of sentences about sets of individuals. The next lowest level will consist of sentences about sets of sets of individuals, and so on. It is then possible to refer to all objects for which a given condition (or predicate) holds only if they are all at the same level or of the same "type."
>
> This solution to Russell's paradox is motivated in large part by the so-called **vicious circle principle**, a principle which, in effect, states that no propositional function can be defined prior to specifying the function's range. In other words, before a function can be defined, one first has to specify exactly those objects to which the function will apply. (For example, before defining the predicate "is a prime number," one first needs to define the range of objects that this predicate might be said to satisfy, namely the set, **N**, of natural numbers.) From this it follows that no function's range will ever be able to include any object defined in terms of the function itself. As a result, propositional functions (along with their corresponding propositions) will end up being arranged in a hierarchy of exactly the kind Russell proposes.
>
> Although Russell first introduced his theory of types in his 1903 ***Principles of Mathematics***, type theory found its mature expression five years later in his 1908 article, ***Mathematical Logic as Based on the Theory of Types***, and in the monumental work he co-authored with **Alfred North Whitehead**, ***Principia Mathematica***."
>
> Russell's type theory thus appears in two versions: the "simple theory" of 1903 and the "ramified theory" of 1908. Both versions have been criticized for being too ad hoc to eliminate the paradox successfully. In addition, even if type theory is successful in eliminating Russell's paradox, it is likely to be ineffective at resolving other, unrelated paradoxes."[1]

From the foregoing account, it may be read that Bertrand Russell had taken functions in the sense of mappings. The mapping concept had emerged along with set theory some years earlier in the history of mathematics. When taking a set e.g. $\{x: x < 5\}$ as a mapping, we are to regard the proposition within the set's name as specifying the rule for the mapping. However a set of horses, for example, does not easily admit to an analysis in which it must be elevated to the world of abstract mathematics. It is one thing to propose such an elevation for certain sets relevant to number theory; quite another to propose it for all sets. Thus it was that Zermelo's solution, in avoiding an abstraction of the set, must have appeared quite attractive when it was published in 1908.

Zermelo's Solution

Zermelo's solution appears to involve a subtle amusement on the old question of which came first, the chicken or the egg. Only instead of asking the question about chicken and egg, Zermelo may have asked the question about set and (for example) horses. Was it three horses and then a set? Or did the set come first? Mathematicians' measure of the counter-intuitive did not rise appreciably at the prospect of the set coming first. Surely it was merely a matter of semantics.

As a result of thinking that the given world was sets, Zermelo proposed to invalidate the widely held view that any property would generate a set. He said "no", we must have a set first, and then any property will subdivide it. But without first being given the parent set, we have no right to assume the existence of a subset based on a generating property. This idea was codified in an axiom which remains canon to this day.

It solves Russell's Paradox (apparently) by questioning our right to generate a set based on the property of not being a member of oneself. The argument in the Paradox calls us to consider the set of all sets which are not members of themselves. This is a set which relies on the generating property of not being a member of oneself: if one has the property, one is in; if one does not have the property, then one is out. According to the Zermelo axiom, the generating property cannot be applied unless existence of a parent set is first granted. A generating property can work only to subdivide a given set. So to escape the logical contradiction, we conclude that no suitable parent set exists. That is to say, the set of all sets does not exist.

If we look around ourselves, we certainly do not see the set of all sets. In fact it would be impossible to apprehend the set of all sets. Thus to rule out its existence did not seem to be terribly onerous on the logic of mathematics.

The drawback with Zermelo's solution arises because the term *all sets* is itself a reference to the set of all sets. How can the latter not exist while existence is implicitly assumed to belong to whatever *all sets* refers to? If existence belongs severally to all the members, existence must belong to the set which contains just those members. To those like myself who object to Zermelo's solution, all the axiom does is to shift the Paradox from one foot to another. It hides the Paradox by lending a cognitive element to set-hood. Cognitive elements are the only elements we can grant or not grant as convention suits us. If we can withdraw set-hood from the set of all sets, then set-hood, as in the quintessential property of a set, must be cognitive element.

New Mathematics seemed to be all about proclaiming that cognitive element. Nevertheless in an historical context, the New Mathematics movement, based on Zermelo's solution to the Paradox, achieved an important distribution of concept. At least in New Zealand, the null set has become widely known and agreed to exist, as certainly as the number 1. As we shall see, this concept is clearly distinguishable from the concept of zero and it will continue to be very useful in the new solution to the Paradox.

VonNeumann-Bernays-Gödel Set Theory

Some astute mathematicians saw the trouble with Zermelo's solution. The sets and classes of everyday life must exist if their members exist. These mathematicians saw there had to be a container for what we refer to as "all sets". So they allowed the class to step in, whence to satisfy. However not wanting to upset the apple cart of the Zermelo proposition, they crafted class theory somewhat as a patch over the logical sore spot in Zermelo's theory. They allowed that the Zermelo axiom would continue, so there would still be no set of all sets, but a class of all sets would be O.K..

Russell's Paradox loomed in the new class. While Zermelo's approach would solve the Paradox for sets, what about solving it for classes? VonNeumann-Bernays-Gödel set theory is known for its answer to this question, namely that classes should not be seen as members at all, not of anything. The class of all classes which are not members of themselves cannot be a member of itself, under this theory, because it is not of a type suitable. To be certain, it has the criteria which defines the class. But under this theory, membership in general must conform to another criterion, that of being a type suitable.

In the wider consideration of the merits of this solution, there has been a coining of the term *proper class* to refer to the classes that cannot be members of anything.

In fact this set theory can be thought of as an attempt to admit a role for plural nomenclature. *All sets* being a plural name, it may be taken to refer to a proper class as opposed to a set. The theory allows that there may be a singular name for a class, as in *the class of all sets*, but if there is a plural name for the class, then the class must be a proper class whether or not there is a singular name as well.

Because the prohibition on membership for classes seems justified in language, with reference to plural nomenclature, the VonNeumann-Bernays-Gödel solution may have appeared as the last word on Russell's Paradox. Unfortunately a strong mathematical intuition says that as a thing comes into existence, so also a property comes into existence, to wit: the property of being the thing. This holds even if the existence is only hypothetical. Whence a thing is already a member of a class, as soon as it would have a singular name. An exception cannot be granted to the proper class.

In effect there are two ways an argument can move, from the observation that classes possess plural names. One can say this prohibits their membership in anything, as in the VonNeumann-Bernays-Gödel solution. On the other hand, one can say that any possession of a formal singular name is accompanied by a requirement for a belonging as member to be allowed. If classes possess singular names, as it appears they must, then it is settled: classes must be allowed to be members of other classes.

The VonNeumann-Bernays-Gödel solution leaves mathematics hanging, as if paused at the door of class theory, uncertain whether to go in. It appears one cannot go in while still preserving the Zermelo proposition. However the class theory applicable to the language of the law does not require a major disenfranchise of the set theoretical canon as widely known. Space for the longevity of most of the results of mathematical set theory is allowed.

Intutionism

The mathematical school of thought known as intuitionism offered a solution to the Paradox through the fundamental intuitionist position on truth. By intuitionist thought, an assertion could be accepted only if there was a constructive, derivational proof for it. From there failing to be any proof for a proposition p, one can hardly deduce that there exists a proof for its propositional negation ~p. So the intuitionists did not accept the law of the excluded middle. My comment above, in the paragraph introducing Russell's Paradox on p. 16, to the effect that there are no other options, does not pass muster by the standards of intuitionism.

Differentiating the Null Set

The concept of the null set is older than Zermelo's solution to Russell's paradox, dating back to when mathematicians decided that all sets would include the null set, a-priori, as a subset. This happened very early in the development of mathematical set theory and it put the set theory on a collision course with the language of the law. To a certain extent, we can treat sets and classes as equivalent. However when it comes to nullity, we must be careful.

Take three horses away from three horses, and we get zero horses, not the null set. Similarly when we reduce a class of three horses to a class of no horses, the class ceases to exist, if indeed existence was ever an issue at all for the class.

Once we have entered into a theory of classes - in such a way as to avoid the error of the VonNeumann-Bernays-Gödel solution - we will not be couching every mathematical assertion in the language of class theory, but rather classes and sets will appear to have complementary domains.

Global Classnames

To address the aim of class theory, as outlined above - to produce a class from any property - we may at first spell out a formula for classnames like *the possible identities of a horse*. Subject to a rule of economy, for any property Ψ, we may say there exists a class for which the primary formal name is *the possible identities of a thing possessing the property Ψ*. The formula allows some variation from this strict wording, indeed encourages grammatically correct shortening, but even after all valid shortening, still we get prefix, indefinite article and name stem in that order.

the possible identities of (prefix)
a (indefinite article)
horse (example stem)

This class-name implicates objects which are merely logically possible as well as relevant objects which actually exist. Thus it may be called a global classname. The possible identities of a blue unicorn make a non-zero class, assuming that blue unicorns are logically possible animals.

We will need this extra breadth because as we turn mathematical law into law about space, the pre-eminent space will be the logical universe - containing both hypothetical space and the physical space of our residence. When we zero in on an evaluation context for a variable, we may arrive at somewhere in hypothetical space as easily as at somewhere in a pressure vessel, or in a patch of earth.

Where reasoning in hypothetical space meets and melds with mathematics, the global classname, as a form, can be distinguished apart and recognised for its potential usefulness in matters of theory. In theory then, *the possible identities of a class* is a valid global classname. Its agreeability may indicate that classes should be contemplated in membership of other classes. Now to answer the question, “What is a class?”, a definition may be proposed along the following lines:

CLASS means any thing for which the primary formal name is plural.

A tighter definition might require the primary formal name to be a plural definite description, such as *the horses* for example, rather than a list. The emphasis on plural definite description would highlight the role of the name stem. The stem usually describes the sort of the class. As we shall see, there are times when knowing the sort is required to judge whether one class includes another and whether certain twists of nomenclature can be applied. However there are other ways of knowing the sort, and a formal list construction method is therefore available. Appropriately constructed lists can be regarded as valid class names as well as definite descriptions.

One ordinarily names a class in anticipation of there being a plurality of members. For if it is known that there is only one member, one usually treats the member for what it is, one hesitates to introduce an unnecessary further classification. However a class name if formal for referring to a plurality of members does not become informal merely through the members reducing in number below two.

Notes

1. Stanford University of Philosophy, Metaphysics Research Lab, CSLI, Stanford University, http://plato.stanford.edu

2. My information about the VonNeumann-Bernays-Gödel solution has largely come from Wikipedia, an internet encyclopaedia.

3. The foregoing is an excerpt from Russell's Paradox - Two Sets, by Russell Z Christensen, ISBN 978-0-473-20648-2, published by Morepork Mystics and Research Team Ltd, New Zealand, 2012.